A Model for Ordinary Levy Motion

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Abstract.

We propose a simple model based on the Gnedenko limit theorem for simulation and studies of the ordinary Levy motion, that is, a random process, whose increments are independent and distributed with a stable probability law. We use the generalized structure function for characterizing anomalous diffusion rate and propose to explore the modified Hurst method for empirical rescaled range analysis. We also find that the structure function being estimated from the ordinary Levy motion sample paths as well as the (ordinary) Hurst method lead to spurious "pseudo-Gaussian" relations.

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By Levy motions, or Levy processes, one designates a class of random functions, which are a natural generalization of the Brownian motions, and whose increments are stably distributed in the sense of P. Levy[1]. Two important subclasses are (i) ordinary Levy motions (oLm's), which generalize the ordinary Brownian motion, or the Wiener process [2], and whose increments are independent, and (ii) fractional Levy motions (fLm's), which generalize the fractional Brownian motions (fBm's) [3] and have an infinite span of interdependence.

The Levy random processes play an important role in different areas of applications, e.g., in economy [4], biology and physiology [5], fractal and multifractal analysis [6], problems of anomalous diffusion [7] etc. In this

paper we provide a simple method for numerical simulation of oLm's and discuss their scaling properties.

At first we show the way to generate random sequence of independent identically distributed (i.i.d.) random variables possessing stable probability law. These variables play the role of increments of the oLm.

We restrict ourselves by symmetric stable law with the stable probability density $p_{\alpha,D}(x)$ and the characteristic function

$$\widehat{p}_{\alpha,D}(k) =< e^{ikx} >= \exp\left(-D\left|k\right|^{\alpha}\right) \tag{1}$$

Here α is the Levy index, $0 < \alpha \leq 2$, and D is a positive parameter. At $\alpha = 1$ and 2 one has the Cauchy and the Gaussian probability laws, respectively. In other cases the symmetric stable laws are not expressed in terms of elementary functions. At $0 < \alpha < 2$ they have power law asymptotic tails *[8]*,

$$p_{\alpha,D}(x) \propto D\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi\left|x\right|^{1+\alpha}}, \quad x \rightarrow \pm\infty \tag{2}$$

Among the methods of generating random sequence with the given probability law *F(x)* the method of inversion seems most simple and effective [9]. However, it is well-known fact that its effectiveness is limited by the laws possessing analytic expressions for F^{-1}, hence, the direct application of the method of inversion to the stable law is not expedient. In this connection, we exploit an important property of stable distributions. Namely, such distributions are limiting for those of properly normalized sums of i.i.d. random variables [10]. To be more concrete, we generate the needed random sequence in two steps. At the first one we generate an "auxiliary" sequence of i.i.d. random variables $\{\xi_j\}$, whose distribution density $F'(x)$ possesses asymptotics having the same power law dependence as the stable density with the Levy index α has, see Eq.(2). However, contrary to the stable law, the function $F(x)$ is chosen as simple as possible in order to get analytic form of F^{-1}. For example,

$$F(x) = \begin{cases} \left[2(1+\left|x\right|^{\alpha})\right]^{-1} & x < 0 \\ 1-\left[2(1+x^{\alpha})\right]^{-1}, & x \geq 0 \end{cases} \tag{3}$$

At the second step the normalized sum

$$X(m) = \frac{1}{am^{1/\alpha}} \sum_{j=1}^{m} \xi, \tag{4}$$

where

$$a = \left(\frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \right)^{1/\alpha} \tag{5}$$

is estimated. According to the Gnedenko theorem on the normal attraction basin of the stable law [10], the distribution of the sum (4) is then converges to the stable law with the characteristic function (1) and D =1. It is reasonable to generate random variables having stable distribution with the unit D, with a consequent rescaling, if necessary. Repeating N times the above procedure, we get a sequence of i.i.d. random variables $\{X_n(m)\}\,, n = 1, ..., N$. In the top of Fig.1 the probability densities $p(x)$ for the members of the sequence $\{X_n(m)\}\,(m = 30)$ are depicted by black points for (a) $\alpha = 1.0$, and (b) $\alpha = 1.5$. The functions $p_{\alpha,1}(x)$ obtained with the inverse Fourier transform, see Eq.(1), are shown by solid lines. In the bottom of Fig.1 the black points depict asymptotics of the same probability densities in log-log scale. The solid lines show the asymptotics given by Eq.(2). It is seen that the Levy index can be estimated with the use of X_n's, which lie outside the peak located around $x = 0$. The examples presented demonstrate a good agreement between the probability densities for the sequences $\{X_n\}$ obtained with the use of the numerical algorithm proposed and the densities of the stable laws.

We would like to stress that a certain merit of the proposed model is its simplicity. It is entirely based on classical formulation of one of the limit theorems and can be easily generalized for the case of asymmetric stable distributions. It is also allows one, after some modifications, to speed up the convergence to the stable law. These problems, however, ought to be the subject of a separate paper. We note, that two schemes were proposed recently, which use the combinations of random number generators [11] and the family of chaotic dynamical systems with broad probability distributions [12], respectively. The former method allows one to generate the sequences with the symmetric laws, whereas the latter allows one to generate also asymmetric ones. The comparison between our scheme and those of Refs.[11, 12] is beyond the scope of our paper. In any case, the proposed method can serve for a further constructing of non-stationary processes and studying of their properties. We proceed to this task below.

With the help of the sequence obtained, the oLm is defined by

$$L_\alpha(t) = \sum_{n=1}^{t} X_n \tag{6}$$

(below we denote time argument as t, τ for the continuous and for the discrete time scales as well; t, τ take positive integer values in the latter case).

In Fig.2 the stationary sequences of independent random variables obtained with the numerical algorithm proposed are depicted by thin lines at 4 different Levy indexes. The thick lines depict the sample paths, or the trajectories, of the oLm's. It is clearly seen that with the Levy index decreasing, the amplitude of the increments increases. The large sparse increments lead to large “jumps” (often named as ”Levy flights”) on the trajectory.

Let us proceed with the properties of self-similarity of the oLm. The characteristic function of the oLm increments is

$$\langle \exp\left[ik(L_\alpha(t+\tau) - L_\alpha(t)\right]\rangle = \exp(-\left|k\right|^\alpha \tau) \tag{7}$$

$(D = 1$ here and below). The increments of the oLm are stationary in a narrow sense,

$$L_\alpha(t_1+\tau) - L_\alpha(t_2+\tau) \stackrel{d}{=} L_\alpha(t_1) - L_\alpha(t_2) \quad , \tag{8}$$

and self-similar with parameter $1/\alpha$, that is, for an arbitrary $h > 0$

$$L_\alpha(t+\tau) - L_\alpha(t) \stackrel{d}{=} \left\{h^{-1/\alpha}\left[L_\alpha(t+h\tau) - L_\alpha(t)\right]\right\} \quad , \tag{9}$$

where $\stackrel{d}{=}$ implies that the two random functions have the same distribution functions.

We consider two corollaries of Eqs.(7) - (9).

1. A ”$1/\alpha$ law” for the generalized structure function (GSF) of the oLm can be stated as follows: for all $0 < \mu < \alpha$ the $1/\mu$-th order root of the GSF is defined by

$$S_\mu^{1/\mu}(\tau,\alpha) = \langle |L_\alpha(t+\tau) - L_\alpha(t)|^\mu\rangle^{1/\mu} = \tau^{1/\alpha} V(\mu;\alpha), \tag{10}$$

where

$$V(\mu;\alpha)=\left\{\int_{-\infty}^{\infty}dx_2\,|x_2|^{\mu}\int_{-\infty}^{\infty}\frac{dx_1}{2\pi}\exp(-ix_1x_2-|x_1|^{\alpha})\right\}^{1/\mu} \tag{11}$$

For the ordinary Brownian motion $\alpha = 2$, and 1/2 law is the indicator of classical (normal) diffusion. Since τ-dependence is not changed with μ varying, then the quantity $S_{\mu}^{1/\mu}(\tau;\alpha)$ at any μ less than α can serve as a measure of anomalous diffusion rate. We remind that the (ordinary) structure function is infinite for $\alpha < 2$.

We study numerically the dependence of the index s in the relation

$$S_{\mu}^{1/\mu}(\tau;\alpha)\propto\tau^{s} \tag{12}$$

vs μ,α. In Fig.3 s vs α is depicted by crosses at fixed $\mu = 1/2$. The $1/\alpha$ curve is shown by primes. One can be convinced himself that the $1/\alpha$ law for the GSF is well confirmed at μ smaller than the smallest Levy index in numerical simulation. For the comparison s vs α is depicted by black points for the structure function, that is, for $\mu = 2$. It is shown that the structure function, being estimated from a finite sample path, lead to the spurious "pseudo-Gaussian" value $s = 1/2$. At the inset s vs μ is depicted for the oLm with $\alpha = 1$. It is shown that $s \cong 1$ at $\mu \le 1$, whereas with μ increasing the deviation from $1/\alpha$ law increases.

The "pseudo-Gaussian" $\tau-$ dependence of the structure function can be explained by the finiteness of sample length taken into account. Indeed, let $X_{\max}$ be the mode of maximum value (that is, the most probable maximum value) for the sequence $\{X_n\}$, which consists from t terms having stable distribution with the Levy index α. It can be easily shown that $X_{\max} \propto t^{1/\alpha}$, and for the variance we get $\langle X^2\rangle \propto t^{2/\alpha-1}$. Therefore, for τ smaller than t (which is the natural condition when estimating the structure function in numerical simulation or at data processing) one gets in the discrete time scale

$$\left\langle (L_\alpha(t+\tau)-L_\alpha(t))^2\right\rangle=\left\langle\left(\sum_{t+1}^{t+\tau}X_n\right)^2\right\rangle\approx\tau\left\langle X_t^2\right\rangle\propto\tau t^{2/\alpha-1}. \tag{13}$$

Thus, the square root of the structure function behaves as $\tau^{1/2}$ for all α's, as it is indeed demonstrated in Fig.3. Furthermore, the value of the structure

function grows with the number t of the terms in the sample path growth. On the contrary, the GSD of the μ-th order, being estimated from a finite sample path at $\mu < \alpha$, does not grow with t growth.

2. A "$1/\alpha$ law" for the span of $L_\alpha(t)$ can be stated as follows:

$$R(\tau) = \sup_{0 \le t \le \tau} L_\alpha(t+\tau) - \inf_{0 \le t \le \tau} L_\alpha(t) \stackrel{d}{=} \tau^{1/\alpha} R(1) \tag{14}$$

For the ordinary Brownian motion $\tau^{-1/2} R(\tau)$ has a distribution independent of τ[13]. In the empirical rescaled range analysis, that is, at experimental data processing or in numerical simulation the span of the random process is divided by the standard deviation (that is, the square root of the second moment) for the sequence of increments, which "smoothes" the variations of the span on the different segments of time series [14]. Such a procedure, called the Hurst method or the method of normalized span, is not satisfactory for the Levy motion because of the infinity of the theoretical value of the standard deviation. Therefore, we propose to modify the Hurst method by exploiting the $1/\alpha$ -th root of the α-th moment instead of standard deviation, that is,

$$\sigma_\alpha = \left(\frac{1}{\tau} \sum_{n=1}^{\tau} |X_n|^\alpha \right)^{1/\alpha} \tag{15}$$

Since it has only weak logarithmic divergence with the number of terms in the sum increasing, then one has

$$\overline{\left(\frac{R(\tau)}{\sigma_\alpha} \right)} \propto \tau^H \quad , \tag{16}$$

where $H \cong 1/\alpha$ is the Hurst index for the oLm with the Levy index α, and the bar denotes averaging over the number of segments (having the length τ) of the sample path.

Fig.4 demonstrates the application of the modified Hurst method to the sample path of the oLm with the Levy index $\alpha = 1$. In Fig.4a the fluctuations of span (thin curve) and those of σ_α (thick curve) are shown for the case, when the total length of the sample is divided into 64 segments, each of $\tau = 16$ lengthwise. Below the variations of the ratio R/σ_α are depicted. It is shown that fluctuations of the ratio is much smaller than those of the span. This circumstance justify the use of the ratio in the empirical analysis. In

Fig.4b the rescaled span vs time interval τ is depicted in log-log scale by black points. The slope of the solid line is equal to $H = 0.9$.

In Fig.5 the H vs α is depicted by crosses, whereas the curve $1/\alpha$ is indicated by primes. The Hurst index obtained from a "traditional" ratio R/σ_2 is shown by black points. It follows from the figure that σ_α only "smoothes" the variations of span, thus leading to the correct value $H = 1/\alpha$. On the contrary, the standard deviation, being used in empirical analysis of the oLm, "suppresses" the variations of the span, thus giving rise to the spurious value $H \approx 0.5$. As in case of the structure function, this "pseudo-Gaussianity" is explained by the finiteness of the sample of the oLm. Indeed, for the segment of the length τ $R(\tau) \propto \tau^{1/\alpha}$, whereas $\sigma_2 \propto \tau^{1/\alpha - 1/2}$, and thus, $R/\sigma_2 \propto \tau^{1/2}$ for all α's. This circumstance allows us to suggest that at estimating the (ordinary) structure function and normalized span from experimental data the "Levy nature" of them can be easily masked. This, in turn, poses an interesting task of developing statistical methods for extracting reliable characteristics from experimental data, for which Levy statistics can be expected from e.g., some physical reasons.

At the end we note that introduction of correlations into the sequences of i.i.d. stably distributed random variables allows us to extend the presented methods on the studies of fractional Levy motions.

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FIGURE CAPTIONS

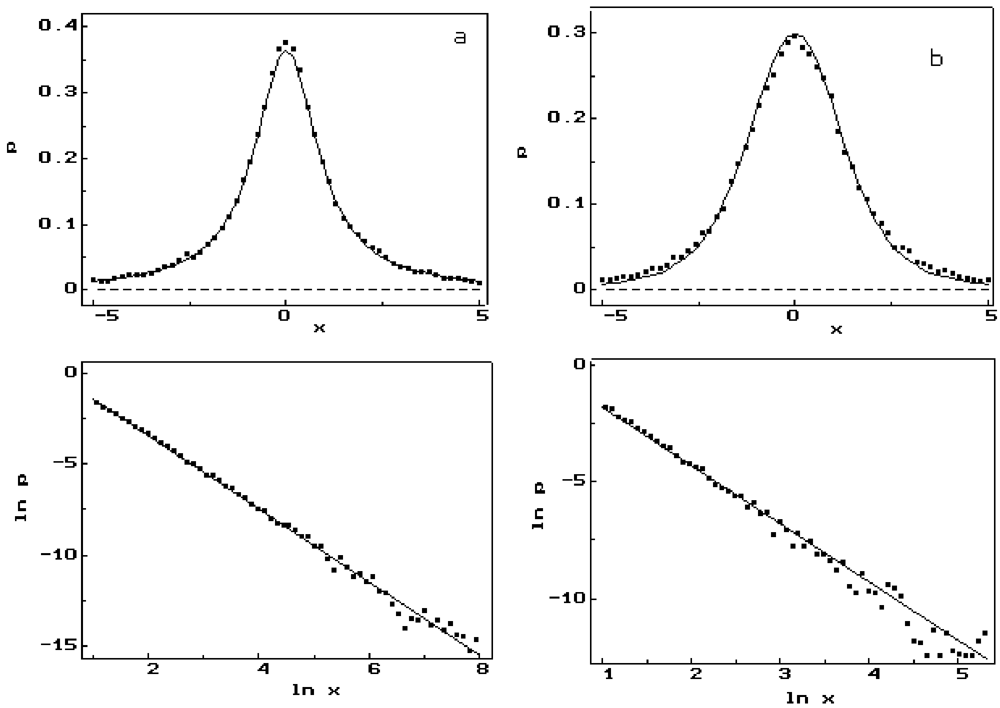
1. Probability densities (above) and their asymptotics (below) are indicated for the sequences of random variables generated with the use of the proposed numerical algorithm at the Levy indexes (a) $\alpha = 1$, and (b) $\alpha = 1.5$. The probability densities and the asymptotics of the stable laws are indicated by solid lines.

2. Stationary sequences (thin lines) and ordinary Levy motion trajectories (thick lines) at the different Levy indexes.

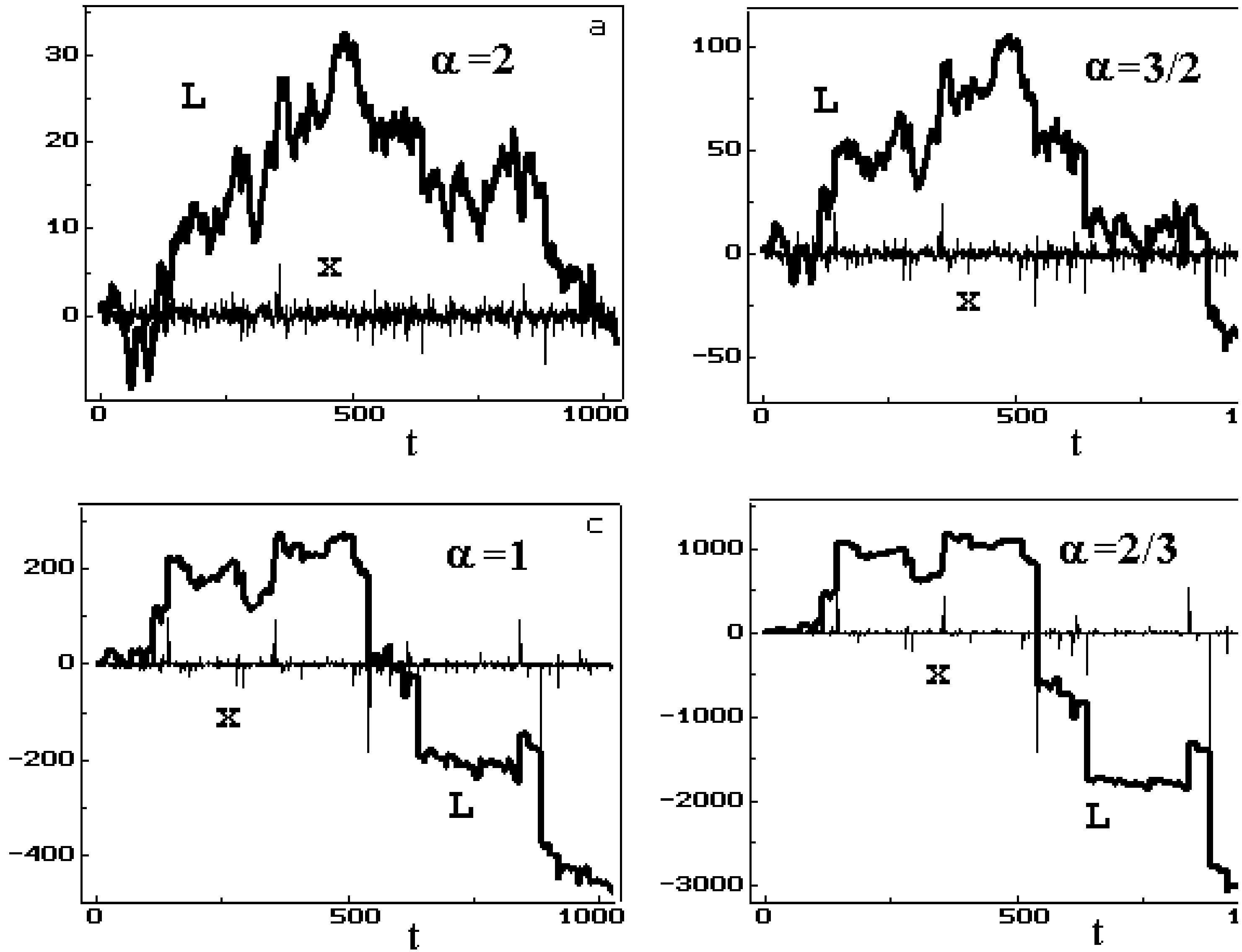
3. Plots of the exponent s in Eq.(12) versus the Levy index α at $\mu = 1/2$ (crosses) and $\mu = 2$ (black points). The $1/\alpha$ curve is depicted by dashed line. At the inset s vs μ at $\alpha = 1$ is shown.

4. (a) The variations of span (thin curve), of the GSD of the α-th order (thick curve) and of their ratio (below) at the different time intervals for the oLm with $\alpha = 1$. (b) Rescaled span vs time interval in log-log scale (black points). Solid line has a slope $H = 0.9$.

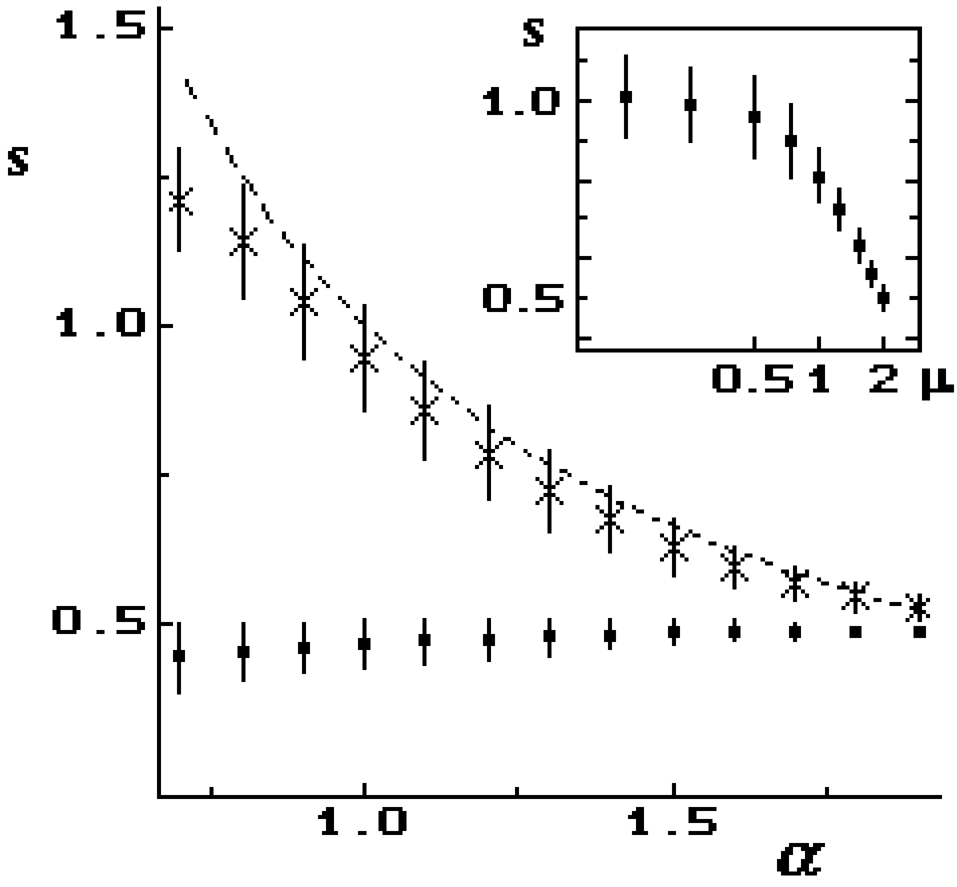
5. Plots of the Hurst exponent H vs α estimated with the use of Eq.(16) (crosses) and with the use of the "traditional" Hurst method (black points). The $1/\alpha$ curve is depicted by dashed line.



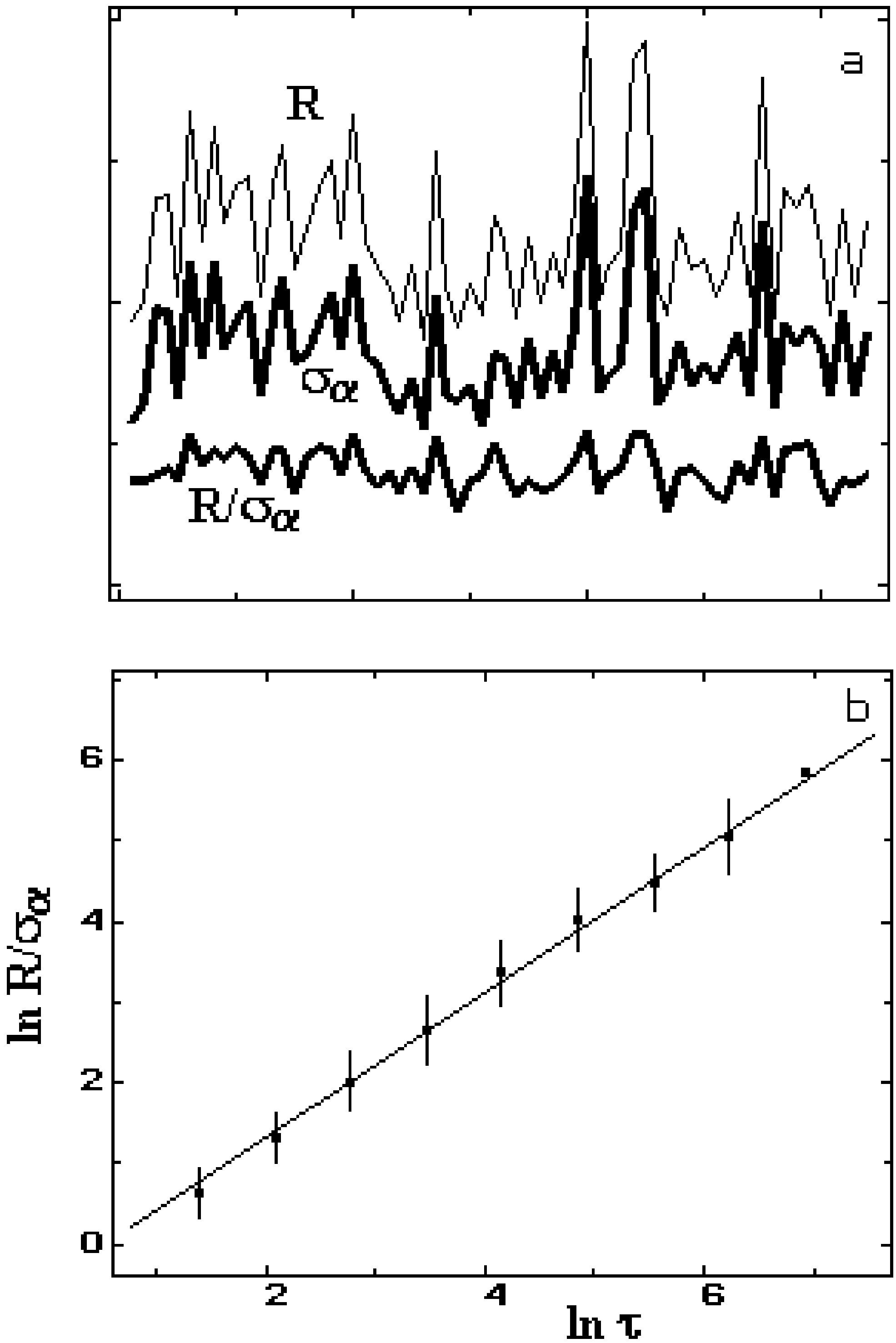
a
b
p
x
ln p
ln x



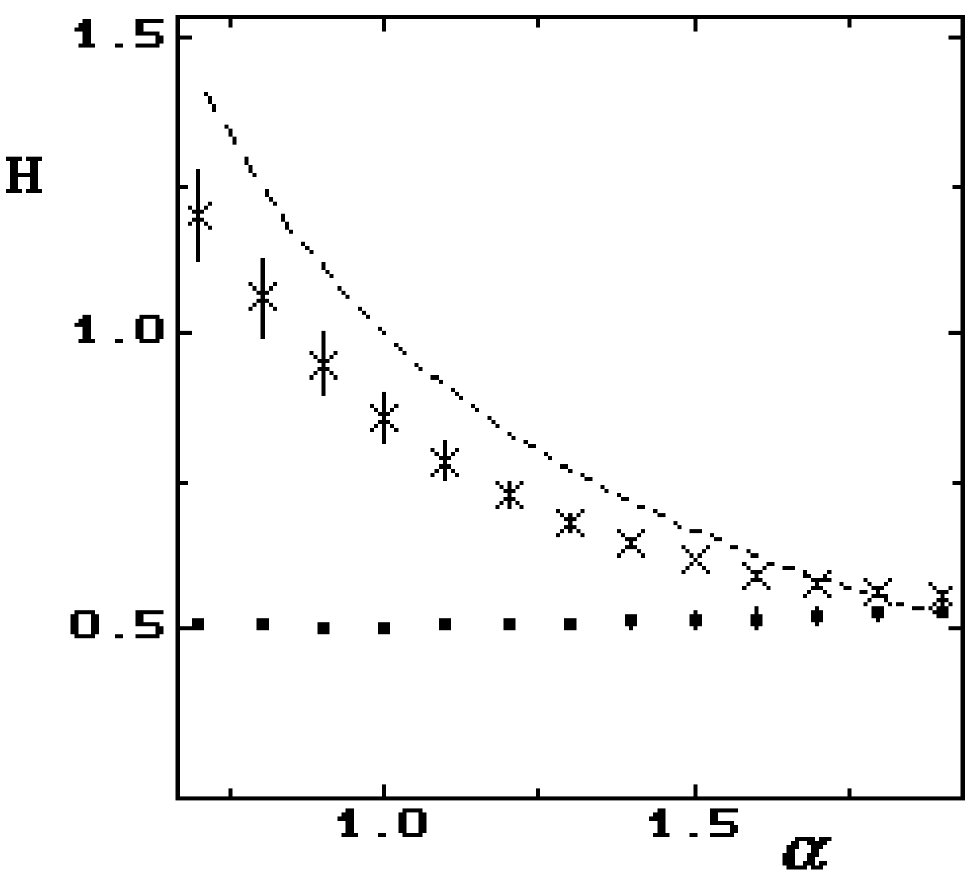
a
α =2
L
x
30
20
10
0
0
500
1000
t
α=3/2
L
x
100
50
0
−50
0
500
1
t
c
α =1
L
x
200
0
−200
−400
0
500
1000
t
α =2/3
L
x
1000
0
−1000
−2000
−3000
0
500
1
t



1.5
S
1.0
0.5
1.0
1.5
α
S
1.0
0.5
0.5 1
2 μ



a
R
σ_α
R/σ_α
b
0
2
4
6
ln R/σ_α
2
4
6
ln τ



1.5
H
1.0
0.5
1.0
1.5
α